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ON FUNCTIONS DEFINED BY AN INFINITE SERIES OF ANALYTIC FUNCTIONS OF A COMPLEX VARIABLE

BY M. B. PORTER

IN the ANNALS OF MATHEMATICS, ser. 2, vol. 3 (1901–02), p. 25, Professor Osgood has proved among others the following theorem.*

If the infinite series

$$f_1(z) + f_2(z) + \cdots$$

whose terms are all functions of z, *single valued and analytic throughout a region* T *of the complex* z*-plane, converges for all values of* z *pertaining to a set of points which is everywhere dense throughout* T *(and which, in particular, may be enumerable), and if furthermore, the relation*

$$|f_1(z) + f_2(z) + \cdots + f_n(z)| \leqq G$$

holds for all points z *of this set (and hence of* T*), and for all values of* n, G *being a positive constant, then the series converges for all values of* z *in* T *and the function* $F(z)$ *defined by the series:*

$$F(z) = f_1(z) + f_2(z) + \cdots$$

is analytic throughout T.

The purpose of this note is to prove by the elementary methods of the Cauchy theory the following generalization of this theorem.

If the infinite series

$$f_1(z) + f_2(z) + \cdots$$

whose terms are all functions of z, *single valued and analytic throughout a region* T *of the complex* z*-plane, converges for all values of* z *pertaining to a point set* (a) *whose limiting points are everywhere dense on a closed rectifiable contour* C *lying inside of* T, *and if the relation*

$$|f_1(z) + f_2(z) + \cdots + f_n(z)| < G$$

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* For another proof of this theorem see a paper by Arzelà in the ANNALS OF MATHEMATICS, ser. 2, vol. 5, p. 51 (1904).

holds for all points z of T and for all values of n, G being a positive constant, then the series converges for all values of z in T and defines an analytic function of z in T.

1. We shall first prove that

$$S(z) = \lim_{n=\infty} S_n(z) = \lim_{n=\infty} [f_1(z) + f_2(z) + \cdots + f_n(z)]$$

converges uniformly for all the points of the contour C. Denote by $\rho \neq 0$ the lower limit of the distances of the points (a) to the boundary of T. We have then,

$$S_n(z) = S_n(a) + b_1(z-a) + b_2(z-a)^2 + \cdots, \qquad |z-a| < \rho,$$

where a denotes any one of the points of the set (a), and where by Cauchy's inequality:

$$|b_n| \leqq \frac{G}{\rho^n}.$$

So that

$$|S_n(z) - S_n(a)| < G \sum_1^\infty \frac{|z-a|^n}{\rho^n} < M\,|z-a|, \quad \begin{cases} n = 1, 2, 3, \cdots, \\ |z-a| < \rho' < \rho, \end{cases}$$

where M is fixed for all the set (a).
Further

$$[1] \qquad |S_{n+p}(z) - S_n(z)| < |S_{n+p}(a) - S_n(a)| + 2M\,|z-a|, \qquad p = 1, 2, 3, \cdots.$$

If now z be a limiting point of (a), a can be so chosen that $2M|z-a| < \frac{\epsilon}{2}$. Let a be so chosen and held fixed, then since $\lim_{n=\infty} S_n(a)$ exists; an N can be determined so that

$$|S_{n+p}(a) - S_n(a)| < \frac{\epsilon}{2}, \qquad n > N,\ p = 1, 2, 3, \cdots.$$

Thus

$$|S_{n+p}(z) - S_n(z)| < \epsilon, \qquad M > N,\ p = 1, 2, 3, \cdots,$$

which shows that $\lim_{n=\infty} S_n(z) = S(z)$ exists (z as above stated denoting a point of C).

That $S(z)$ converges uniformly to its limit is shown thus; let $\bar{z}$ denote a point of C different from z such that

$$|z - \bar{z}| < \rho' < \rho ;$$

then precisely as in (1) we show that

$$|S_{n+p}(z) - S_{n+p}(z)| < |S_{n+p}(\bar{z}) - S_n(\bar{z})| + 2M|z - \bar{z}|, \qquad p = 1, 2, 3, \cdots .$$

If now we choose $\bar{z}$ so that

$$2M|z - \bar{z}| < \frac{\epsilon}{2},$$

we get

$$|S_{n+p}(\bar{z}) - S_n(\bar{z})| < \epsilon, \qquad n > N, \quad p = 1, 2, 3, \cdots ,$$

provided

$$|z - \bar{z}| < \frac{\epsilon}{4M},$$

which proves that $S(z)$ converges uniformly on C.

2. We shall now prove that $S(z)$ *converges and converges uniformly in any region lying inside of C and therefore by a well known theorem represents an analytic function in such region.*

To do this we shall need the following lemma, to which, though doubtless well known, I can give no reference in the literature.

LEMMA. If a series of functions

$$f_1(z) + f_2(z) + \cdots ,$$

single valued and analytic inside a region S and continuous along the boundary C of S, supposed rectifiable, converges uniformly along C, then

$$f_1(z) + f_2(z) + \cdots$$

converges uniformly and represents an analytic function of z in any region lying inside of C.

Proof. Let t denote the affix of any point on C, L any region inside of C such that the minimum distance of the points of L to the contour C is $\sigma \neq 0$, and z any point of L; then by Cauchy's integral formula we have:

$$S_n(z) = \frac{1}{2\pi i} \int_C \frac{S_n(t)\,dt}{t - z},$$

where

$$S_n(z) = \sum_1^n f_i(z) ;$$

also
$$|S_{n+p}(z) - S_n(z)| = \frac{1}{2\pi}\left|\int_C \frac{S_{n+p}(t) - S_n(t)}{t - z}\,dt\right|$$
$$< \frac{1}{2\pi}\frac{\epsilon}{\sigma}C < \epsilon',$$

where C is the length of the contour C, and where, since $S(t) = \lim_{n=\infty} S_n(t)$ is uniformly convergent, a number N can be found such that

$$|S_{n+p}(t) - S_n(t)| < \epsilon, \qquad n > N.$$

Thus ϵ' can be made as small as we please for all points of L at once by a suitable choice of N, which demonstrates the Lemma.

The application of this lemma shows that the theorem stated at the beginning of §2 is true, and applying a well known theorem of Stieltje's,* $S(z)$ is seen to be analytic throughout the region T.

An examination of the foregoing proof shows that it is valid if the condition

$$|\sum_1^n f_i(z)| < G$$

is merely imposed for the part of T lying outside of a region of the type styled L in §2.

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* See *Ann. de la Faculté des Sciences de Toulouse*, vol. 8 (1894), p. J. 56, or Professor Osgood's paper above cited.